J. Fluid Mech. (1975), *vol.* 70, *part* 1, *pp.* 17–36
Printed in Great Britain

Fluid shielding of low frequency convected sources by arbitrary jets

By T. F. BALSA
General Electric Company, Corporate Research and Development,
P.O. Box 43, Schenectady, New York 12301

(Received 3 July 1974)

A low frequency asymptotic theory is proposed for the shielding of noise by jets of arbitrary cross-section. The results of the theory provide a qualitative explanation for the appearance of the quiet and noisy planes of a slot jet. The arguments in favour of this explanation are derived from a model problem in which a pulsating mass source is convecting along the axis of an infinitely long column of fluid of arbitrary cross-section. The jet velocity is represented by a uniform velocity profile (i.e. slug flow). The method of matched asymptotic expansions is applied to derive expressions for the acoustic pressure and the radiative power of the source.

The solution for the elliptic jet indicates that the radiative power in the horizontal plane (containing the major axis) is less than that in the vertical plane (containing the minor axis). This difference in power varies with source Strouhal number and jet Mach number. The effects of jet temperature are also included in the analysis. The theoretical results are in good qualitative agreement with experimental findings for slot nozzles. The theory indicates that the noise shielding offered by jets is negligible at low frequencies and low Mach numbers.

1. Introduction

A completely rational and satisfactory theory of jet noise has escaped the concentrated efforts of scientists. No theoretical model can, at the moment, predict the complete acoustic characteristics of complex nozzle configurations.

One serious difficulty lies in the description of the noise source itself. A great step in this direction has been taken by Lighthill (1952), who identified the source as convecting and randomly fluctuating turbulent eddies. Although the qualitative aspects of this picture appear to be correct, to date there are no mathematical models that describe the quantitative aspects of the physics.

Another serious conceptual difficulty arises in the description and interpretation of the fluctuating pressure p itself. It is undoubtedly true that, very far from the jet, the perturbation pressure is 'acoustic' in the sense that it obeys a linear hyperbolic equation. To solve this equation uniquely, suitable initial and boundary conditions must be provided. Since the linear perturbation equation is valid only far away from the jet, the boundary conditions must be given on a large surface enclosing the jet. The governing equation can then be used to continue the pressure to another large surface. Unfortunately no such boundary conditions can be prescribed theoretically at present, so the problem for the acoustic pressure cannot be solved rigorously.

One very approximate and physically meaningful way around this difficulty is to assume that a perturbation equation holds everywhere in the fluid, including the jet, and that the noise sources act as forcing terms for this equation. This is Lighthill's acoustic analogy.

The Navier–Stokes equations, which are believed to describe jet noise the most accurately, are of course known. It is possible to *rearrange* these equations into the form

$$L\mathbf{w} = \mathbf{f} + \mathbf{g}, \tag{1a}$$

where L is a linear hyperbolic operator, $\mathbf{w}$ is the dependent variable (generally a vector as written here), $\mathbf{f}$ is the noise source and $\mathbf{g}$ is a remainder, usually neglected. The only restriction on (1a) is that far away from the jet $\mathbf{f}, \mathbf{g} \to 0$ and $L \sim \Box$, where $\Box$ is the classical wave operator and the tilde denotes equivalence. One minor assumption implied by the last sentence is that the medium external to the jet is perfectly homogeneous and at rest. It turns out that the rearrangement of the Navier–Stokes equations into the form (1a) is not unique. Thus L, $\mathbf{f}$ and $\mathbf{g}$ are somewhat arbitrary: intuition and physical insight must determine any two of L, $\mathbf{f}$ and $\mathbf{g}$. Lighthill (1952) assumed that $L \equiv \Box$ and $\mathbf{g} = 0$ (with $\mathbf{w}$ replaced by $\tilde{\rho}$) and showed that $\mathbf{f}$ must be of the form of a double space derivative. One undesirable consequence of the Lighthill formulation is that $\mathbf{f}$ depends on the density $\tilde{\rho}$. Perhaps a more satisfactory description of the acoustic pressure is offered by Phillips (1960) and Lilley (1972). Lilley's equation, valid for unidirectional flows, is

$$\frac{D}{Dt}\left(\frac{D^2 p}{Dt^2} - \tilde{c}^2 \Delta p - \frac{d\tilde{c}}{dx_3}\frac{\partial p}{\partial x_3}\right) + 2\tilde{c}^2 \frac{d\tilde{U}}{dx_3}\frac{\partial^2 p}{\partial x_1 \partial x_3} = \tilde{c}^2 \tilde{\rho}\frac{D}{Dt}\frac{\partial^2}{\partial x_i \partial x_j}[u_i' u_j' - \overline{u_i' u_j'}] + \dots, \tag{1b}$$

where $D/Dt = \partial/\partial t + \tilde{U}\,\partial/\partial x_1$. In this equation the undisturbed jet velocity $\tilde{U}$ and jet sound speed $\tilde{c}$ are assumed to be functions of the co-ordinate x_3 only. The jet axis is along the x_1 axis and $\tilde{\rho} = \tilde{\rho}(x_3)$ is the undisturbed density. u_i' is the randomly fluctuating part of the ith velocity component and the overbar denotes a time average. Observe that we have omitted the shear noise term in (1b), time is denoted by t and $\mathbf{x} = (x_1, x_2, x_3)$ is a fixed co-ordinate system.

The right-hand side of (1b) has been essentially identified by Lighthill as an acoustic quadrupole. Its left-hand side accounts for refraction (Ribner 1960, 1962) and fluid shielding or shrouding (Ribner 1960; Powell 1960; Csanady 1966). It was Mani (1972) who first solved a model problem to quantify the effects of fluid shielding on total radiated power. It now appears that the three most important physical aspects of jet noise are source convection, refraction and fluid shielding, although it is generally not fruitful to view refraction and shielding separately. Furthermore, recent work of Mani (1974) indicates that fluid shrouding effects can, at least partially, explain the variation of the jet density exponent of hot jets.

Convective amplification, as given by Lighthill, is frequency independent.

On the other hand, the combined convection–fluid-shielding factor of Mani is strongly frequency dependent. In fact, at very high frequencies, Mani (1972) found that convective amplification altogether disappears. A purely physical explanation for this has been given by Ribner (1960) and Csanady (1966) and is known as the Ribner–Csanady co-moving fluid hypothesis.

The work of Mani concentrates on the shielding offered by circular jets. It is natural to extend his results to jets of arbitrary cross-section. The purpose of this paper is to show how such an extension can be carried out at *low frequencies* and to assess the effect of non-circularity on shielding. It should be borne in mind, however, that shielding is most effective at *high frequencies*, so that a low frequency theory can, at most, indicate trends as the frequency is increased. (By comparing our approximate results with Mani's exact results for a circular jet, we establish an upper limit for the validity of our low frequency theory. This upper limit is reasonably high, so that a low frequency theory can provide useful information even for moderately high frequencies.)

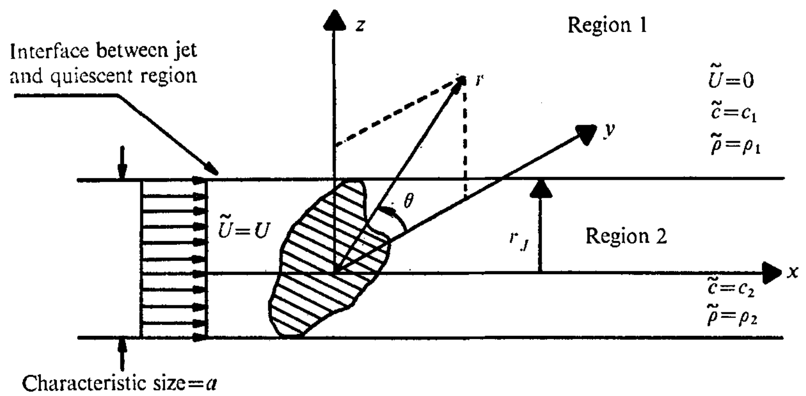

FIGURE 1. Idealized geometry of an arbitrary jet.

Considerable experimental evidence (Olsen, Gutierrez & Dorsch 1973; Hoch & Hawkins 1973) indicates that the noise characteristics of non-circular jets are a strong function of the specific plane of measurement: not only that the spatial distributions of the pressure level are different but also that the power radiated per unit polar angle varies with the angle itself (figure 1). The last remark suggests an easy and reasonably efficient method for reducing jet noise in certain directions.

A number of qualitative arguments, usually involving turbulent mixing, provide a possible explanation for this selective appearance of 'quiet' and 'noisy' planes. Recently, Crighton (1973) has advanced a theoretical stability argument for an incompressible elliptic jet. His results show that the instability is more pronounced along the minor axis than along the major one. These results certainly support the concept that the noise level of a jet can be modified by the stability characteristics of the mean flow.†

† The author is grateful to one of the referees for pointing out that the crackle phenomenon discussed by Ffowcs Williams (1973) is indeed caused by the instability of the flow.

In this paper we show that a plausible explanation for the appearance of the quiet and noisy planes is acoustic shielding. The arguments in favour of this explanation are drawn from the results of a model problem.

In real, high Mach number jets, of course, both the aerodynamic and acoustic effects are present. Through carefully designed fundamental experiments it will be possible to determine which of these effects is more significant. Once this has been determined, a real step will have been taken towards the understanding of jet noise.

Our formulation of the problem parallels that of Mani (1972) quite closely. We assume that the jet velocity profile can be represented sufficiently accurately by a uniform velocity profile (i.e. by a plug-flow profile). Certainly at low frequencies the precise form of the radial gradients of the mean flow is not important. Slowly varying axial gradients could be handled by an extension of our asymptotic theory. Our basic idea is to divide the flow regime, in the sense of matched asymptotic expansions, into two regions, one near and the other far from the jet. In the inner region the axial gradients may be neglected since the appropriate length scale is the diameter of the jet. On the other hand, the pressure in the outer region obeys the classical wave equation, so that the velocity (or its gradient) does not enter the outer solution. Clearly then, the basic approach used in this paper is applicable to slow axial variations in the mean flow provided that at each point x_1 the local velocity $\tilde{U}(x_1)$ is used.†

Our governing equation is Lilley's equation with $\tilde{U}$ = constant and $\tilde{c}$ = constant but with $\tilde{U}$ and $\tilde{c}$ discontinuous across the jet interface. The terms $d\tilde{c}/dx_3$ and $d\tilde{U}/dx_3$ are dropped from (1*b*) (these are identically zero except at the jet boundary) and the pressure and particle displacement are required to be continuous across the interface. It is known that solutions to the above problem generally exhibit the Kelvin–Helmholtz instability. We shall briefly touch upon this topic in another section.

2. Formulation of the model problem

We begin by assuming that the acoustic field obeys a linear wave equation of the form

$$\left(\frac{\partial}{\partial t}+\tilde{U}\frac{\partial}{\partial x}\right)^2\Phi-\tilde{c}^2\Phi_{xx}-\tilde{c}^2\Delta\Phi = D, \qquad (2)$$

where Φ is the velocity potential, D is the disturbance that generates the acoustic field, t is time, x is an axial co-ordinate, along which the fluid velocity is $\tilde{U}+\Phi_x$, where $\tilde{U}$ = constant.‡ The undisturbed speed of sound is represented by the constant $\tilde{c}$ and $\Delta = \nabla^2$ is the Laplacian in the transverse co-ordinates. Physically, (2) represents the propagation of an acoustic disturbance whose fluid velocity is $(\Phi_x, \nabla\Phi)$ in a uniform stream of speed $\tilde{U}$.

As are all partial differential equations, (2) is solved in a specific space domain.

† This observation is perfectly analogous to classical slender-body theory, in which the body shape varies slowly in the axial direction.

‡ Note that $\tilde{U}$, $\tilde{c}$, ..., etc., are assumed to be constant over each cross-section.

This domain is illustrated in figure 1 together with the appropriate values of $\tilde{U}$, $\tilde{c}$ and the undisturbed density $\tilde{\rho}$. Thus, for $r < r_J$, $\tilde{U} = U$, $\tilde{c} = c_2$ and $\tilde{\rho} = \rho_2$; for $r > r_J$, $\tilde{U} = 0$, $\tilde{c} = c_1$ and $\tilde{\rho} = \rho_1$, where

$$r = r_J(\theta), \quad 0 \leqslant \theta < 2\pi, \tag{3}$$

is the equation of a doubly infinite cylinder. We assume that $2r_J < a$, where a is a given constant.

It remains to model the source term D. Here we are guided by the desire to maintain simplicity and the success of the work of Mani (1972). We assume

$$D = \exp(-i\omega_0 t)\,\delta(x - Ut)\,f(r,\theta)/a^2, \tag{4}$$

where ω_0 is a given constant and $f(r,\theta)$ is a given function such that $f = 0$ for $r > r_J$. Physically, the source term represented by (4) is a harmonically oscillating disturbance of frequency ω_0 concentrated at $x = Ut$. The source is, of course, convecting at speed U in the $+x$ direction. The assumption that the source convection speed is the same as the jet speed is made for simplicity.

Across the interface between the jet and the quiescent region we require continuity of the perturbation pressure p and the particle displacement η. The latter assumption implies negligible mixing between the jet and the surrounding medium. The expression for the perturbation pressure is most easily derived from the linearized x-momentum equation, and is given by

$$p = -\tilde{\rho}(\Phi_t + \tilde{U}\Phi_x), \tag{5a}$$

where $\tilde{\rho}$ is the undisturbed density in the appropriate region. The particle displacement η obeys

$$\partial\Phi/\partial\eta = \eta_t + \tilde{U}\eta_x, \tag{5b}$$

where $\partial\Phi/\partial\eta$ is the velocity normal to the mean location of the interface between the jet and the quiescent region.

We remark that the undisturbed static pressure is also continuous across the jet boundary. This implies that, for given ambient conditions, the jet density or speed of sound (i.e. temperature) determines uniquely the undisturbed thermodynamic state of the jet.

Our governing equation (2) is hyperbolic and requires initial conditions for uniqueness. These can easily be provided (for example, $\Phi = \partial\Phi/\partial t = 0$ at $t = 0$). However, in the present context we are interested in the long-time solution (as we essentially follow the disturbance), and the initial conditions have negligible effect on this.

With the above preliminary remarks in mind, we extract the time and x dependence in (2) through Fourier transforms. Let us define

$$\Phi^* = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} \exp(i\omega t)\,\Phi\,dt, \quad \Phi = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} \exp(-i\omega t)\,\Phi^*\,d\omega \tag{6a, b}$$

and†

$$\Phi^* = \overline{\Phi^*}\exp[i(\omega - \omega_0)\,x/U]. \tag{6c}$$

† After performing the Fourier transform in time, it is clear that the x dependence of the solution must be of the form given by (6c). Note that a Fourier transform in x also leads to (6c) after inversion.

Using these transforms the resultant equation for $\overline{\Phi^*}$ is given in the *still-air region* by

$$\Delta\overline{\Phi^*} + k_0^2(K_1^+)^2\,\overline{\Phi^*} = 0 \tag{7a}$$

and in the *jet region* by

$$\Delta\overline{\Phi^*} + k_0^2(K_2^+)^2\,\overline{\Phi^*} = \frac{-1}{(2\pi)^{\frac{1}{2}}c_2^2\,U}f(r,\theta)/a^2 \tag{7b}$$

or

$$\Delta\overline{\Phi^*} - k_0^2(K_2^-)^2\,\overline{\Phi^*} = \frac{-1}{(2\pi)^{\frac{1}{2}}c_2^2\,U}f(r,\theta)/a^2, \tag{7c}$$

where $k_0 = \omega_0/c_1$. The propagation constants K_1^+, K_2^+ and K_2^- are given by

$$(K_1^+)^2 = \kappa^2 - (\kappa-1)^2/M^2, \tag{8a}$$

$$(K_2^+)^2 = \Gamma_{12}\rho_{21} - (\kappa-1)^2/M^2, \tag{8b}$$

$$(K_2^-)^2 = (\kappa-1)^2/M^2 - \Gamma_{12}\rho_{21}, \tag{8c}$$

where $\kappa = \omega/\omega_0$, $M = U/c_1 < 1$, $\rho_{21} = \rho_2/\rho_1$ and $\Gamma_{12} = \Gamma_1/\Gamma_2$, with

$$\Gamma = (1-\mathscr{R}/c_p)^{-1}. \tag{8d}$$

$\mathscr{R}$ denotes the gas constant and c_p is the specific heat at constant pressure. An additional assumption made in deriving (8*b*, *c*) is that the gas is thermally perfect (but not calorically perfect).

We are interested only in the case $(K_1^+)^2 \geqslant 0$ (otherwise the far-field solution dies off exponentially in the transverse variable), that is, $(1+M)^{-1} \leqslant \kappa \leqslant (1-M)^{-1}$. When $(K_2^+)^2 \geqslant 0$, that is, $(1+M)^{-1} \leqslant \kappa \leqslant 1+M(\Gamma_{12}\rho_{21})^{\frac{1}{2}}$, equation (7*b*) holds; otherwise (7*c*) holds.

The behaviour of K_1^+ and $K_2^\pm$ is shown in figures 2 (*a*) and (*b*) for several values of M. In these figures it is assumed that $\Gamma_{12}\rho_{21} = 1$.

To complete the present framework for the formulation of the problem, we extract the t and x dependence of the matching conditions across the interface. In the still-air region we find that the transforms of (5*a*, *b*) are

$$\overline{p^*} = i\rho_1\omega\,\overline{\Phi^*}, \quad \overline{\eta^*} = \frac{i}{\omega}\frac{\partial\overline{\Phi^*}}{\partial n} \tag{9a, b}$$

and in the jet region

$$\overline{p^*} = i\rho_2\omega_0\,\overline{\Phi^*}, \quad \overline{\eta^*} = \frac{i}{\omega_0}\frac{\partial\overline{\Phi^*}}{\partial n}. \tag{9c, d}$$

Thus the canonical problem to be solved is

$$\Delta\phi \pm k_0^2K^2\phi = f(r,\theta)/a^2, \tag{10a}$$

with matching conditions

$$\kappa\phi_1 = \rho_{21}\phi_2, \quad (\partial\phi/\partial n)_1 = \kappa(\partial\phi/\partial n)_2 \quad \text{on} \quad r = r_J, \tag{10b, c}$$

where K^2 stands for $(K_1^+)^2$, $(K_2^+)^2$ or $(K_2^-)^2$. Note that $\phi = -(2\pi)^{\frac{1}{2}}c_2^2\,U\overline{\Phi^*}$ and ϕ_2 denotes the solution in the jet while ϕ_1 applies in the ambient medium.

Observe that, after taking the convective derivative $\partial/\partial t+\bar{U}\,\partial/\partial x$ of (2) and (4), we arrive at

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2 p-c_2^2p_{xx}-c_2^2\Delta p = i\rho_2\omega_0\exp\left(-i\omega_0t\right)\delta(x-Ut)f(r,\theta)/a^2 \quad (11a)$$

and

$$p_{tt}-c_1^2p_{xx}-c_1^2\Delta p = 0 \quad (11b)$$

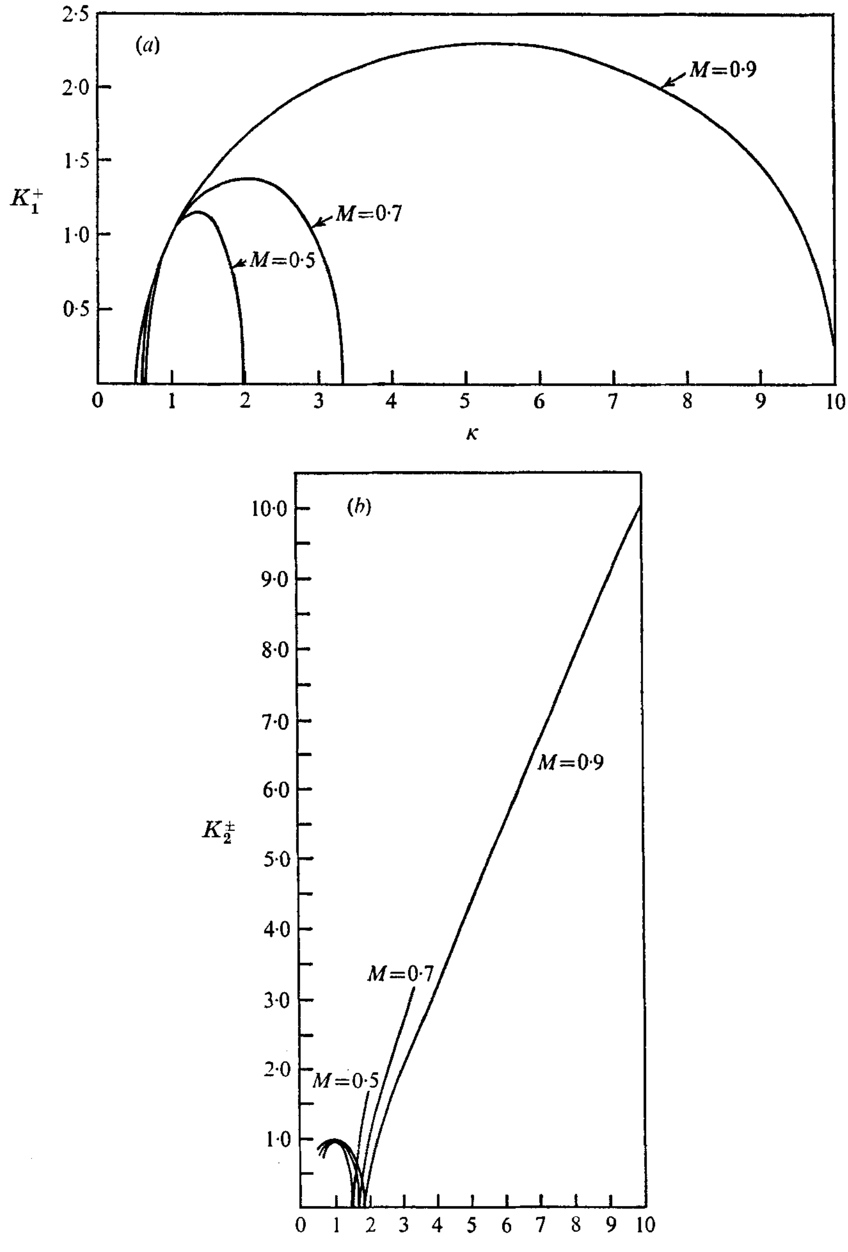

FIGURE 2. (*a*) Free-space propagation constant K_1^+ and (*b*) jet propagation constant $K_2^\pm$ *vs*. frequency κ.

in the jet and quiescent regions respectively. Therefore the convective derivative of the solution to (10a) provides the solution of (11). The left side of (11) is essentially that of Lilley's equation with $\tilde{U}$ and $\tilde{c}$ constant over each cross-section. Also note that differentiating the solution of (11) with respect to x once (twice) yields the solution for a generalized axial pressure dipole (quadrupole). Furthermore, by specializing $f(r,\theta)$ to suitable singular distributions, the solution for any point dipole or quadrupole field can be readily obtained. As an example, let $f(r,\theta) = \delta(r-r_0)\,\delta(\theta-\theta_0)/r$, where r_0 and θ_0 are parameters. Differentiating this solution of (11) with respect to r_0 and θ_0 yields the solution for an $r-\theta$ quadrupole.

Of course, the last remark merely states that, if the source solution (i.e. the Green's function) to (10a) is known, the solution for an arbitrary forcing function $f(r,\theta)$ can be written down by superposition. Since our jet has a completely arbitrary cross-section, we may place the origin of our co-ordinate system at the source location. Thus our canonical problem can be replaced by

$$\Delta\phi \pm k_0^2 K^2 \phi = \delta(r)/r \tag{12}$$

without any loss of generality. Boundary conditions (10b, c) are of course retained.

3. Expansion of the inner and outer solutions

We begin by introducing an inner variable $\tilde{r} = r/a$ and rewriting (12) as

$$\frac{\partial^2\phi}{\partial\tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial\phi}{\partial\tilde{r}} + \frac{1}{\tilde{r}^2}\frac{\partial^2\phi}{\partial\theta^2} + \epsilon^2K^2\phi = \tilde{\Delta}\phi + \epsilon^2K^2\phi = \frac{\delta(\tilde{r})}{\tilde{r}}, \tag{13}$$

where a denotes the characteristic size of the jet and $\epsilon = k_0 a$. We wish to find an asymptotic solution of (13) as $\epsilon \to 0$. For the time being we assume that K^2 is about unity and later show how to improve the asymptotic solution when K^2 is considerably larger. The asymptotic limit $\epsilon \to 0$ clearly corresponds to a low frequency solution of (11).

We consider inner gauge functions $\delta_{\nu\mu}^{(i)}(\epsilon) = (\frac{1}{2}\epsilon)^{2\nu}\log^{\mu}(\frac{1}{2}\epsilon)$ and expand the inner solution $\phi^{(i)}$ outside the jet as

$$\phi^{(i)} = \sum_{\nu,\mu} \delta_{\nu\mu}^{(i)}(\epsilon)\,\phi_{\nu\mu}^{(i)}, \quad \nu = 0,1,\ldots, \quad \mu = 0,1,\ldots,\nu+1. \tag{14}$$

The form of the inner gauge functions is suggested by the results of classical slender-body theory (Germain 1967). The theory of supersonic flow over a slender body is formally equivalent to the low frequency theory of a pulsating body (Miles 1953; Landau & Lifshitz 1959, p. 283). Thus it is not surprising that the same gauge functions arise in the present problem as did in classical slender-body theory. After substituting (14) into (13) and collecting like gauge functions in ϵ, we arrive at

$$\tilde{\Delta}\phi_{00}^{(i)} = 0 \quad \text{at order } 1 \quad (\text{'lowest' order solution}), \tag{15a}$$

$$\tilde{\Delta}\phi_{01}^{(i)} = 0 \quad \text{at order} \quad \log(\tfrac{1}{2}\epsilon), \tag{15b}$$

$$\tilde{\Delta}\phi_{12}^{(i)} = 0 \quad \text{at order} \quad (\tfrac{1}{2}\epsilon)^2\log^2(\tfrac{1}{2}\epsilon), \tag{15c}$$

$$\tilde{\Delta}\phi_{11}^{(i)} + 4K^2\phi_{01}^{(i)} = 0 \quad \text{at order} \quad (\tfrac{1}{2}\epsilon)^2\log(\tfrac{1}{2}\epsilon), \tag{15d}$$

$$\tilde{\Delta}\phi_{10}^{(i)} + 4K^2\phi_{00}^{(i)} = 0 \quad \text{at order} \quad (\tfrac{1}{2}\epsilon)^2, \quad \text{etc.} \tag{15e}$$

Thus the sequence of inner solutions satisfies Poisson's (or Laplace's) equation. The inhomogeneous terms are either given or are known from the lower-order solutions.

We next introduce an outer variable $R = k_0 r = \epsilon\tilde{r}$ and a sequence of outer gauge functions $\delta^{(o)}_{\nu\mu}(\epsilon)$ (to be determined by matching asymptotically the outer solution to the inner solution). We rewrite (12) as

$$\frac{\partial^2\phi}{\partial R^2}+\frac{1}{R}\frac{\partial\phi}{\partial R}+\frac{1}{R^2}\frac{\partial^2\phi}{\delta\theta^2}+K^2\phi = \Box\phi = 0, \quad R > 0. \tag{16}$$

If we represent the outer solution by

$$\phi^{(o)} = \sum_{\nu,\,\mu} \delta^{(o)}_{\nu\mu}(\epsilon)\,\phi^{(o)}_{\nu\mu} \tag{17a}$$

then each term in the expansion obeys

$$\Box\phi^{(o)}_{\nu\mu} = 0. \tag{17b}$$

Clearly the sequence of outer solutions obeys the homogeneous Helmholtz equation.

4. Inner and outer solutions and results of asymptotic matching

We first regroup the inner and outer gauge functions $\delta^{(i)}_{\nu\mu}$ and $\delta^{(o)}_{\nu\mu}$ to form a three-term asymptotic sequence to the required order of accuracy. Let

$$E = 1+\log\left(\tfrac{1}{2}\epsilon\right)+\log^2\left(\tfrac{1}{2}\epsilon\right),$$

then the first three terms of this new gauge sequence are

$$\delta_1(\epsilon) = E, \quad \delta_2(\epsilon) = \left(\tfrac{1}{2}\epsilon\right)E, \quad \delta_3(\epsilon) = \left(\tfrac{1}{2}\epsilon\right)^2 E. \tag{18a–c}$$

The matching is done, term by term, for the coefficient of each δ_n $(n = 1, 2, 3)$. Such regrouping is absolutely essential for the success of the asymptotic matching principle as given by Van Dyke (1964, p. 90). The reasons for regrouping are thoroughly discussed in a series of papers by Fraenkel (1969). The point here is that a *given* function can be expanded in several different sequences of gauge functions $\delta^{(i)}_{\nu\mu}$ and $\delta^{(o)}_{\mu\nu}$. Since the function is given, there can be no question of the validity of any of these expansions. However, not all of these expansions satisfy the asymptotic matching principle of Van Dyke (1964). In other words, it is possible to have a situation in which the expansion is correct, but the inner and outer solutions cannot be matched. Such is the case in the present analysis if term-by-term matching is required for the gauge sequences $\delta^{(i)}_{\nu\mu}$ and $\delta^{(o)}_{\mu\nu}$.

However, if the regrouping is done according to (18), the inner and outer solutions can be matched. For example, the matching must be done for the coefficient of the term $1+\log\left(\tfrac{1}{2}\epsilon\right)+\log^2\left(\tfrac{1}{2}\epsilon\right)$ rather than individually for the coefficients of 1, $\log\left(\tfrac{1}{2}\epsilon\right)$ and $\log^2\left(\tfrac{1}{2}\epsilon\right)$. After a step-by-step application of the asymptotic matching principle, we find that in the inner region (but outside the jet)

$$\phi_{00}^{(i)} = A_0^{(1)} + B_0^{(1)} \log \tilde{r} + \sum_{n=1}^{\infty} \tilde{r}^{-n}(C_n^{(1)} \cos n\theta + D_n^{(1)} \sin n\theta), \tag{19a}$$

$$\phi_{01}^{(i)} = A_0^{(2)}, \quad \phi_{12}^{(i)} = A_0^{(3)}, \tag{19b, c}$$

$$\phi_{11}^{(i)} = A_0^{(4)} + B_0^{(4)} \log \tilde{r} + \sum_{n=1}^{\infty} \tilde{r}^{-n}(C_n^{(4)} \cos n\theta + D_n^{(4)} \sin n\theta) - 2K^2\tilde{r}(C_1^{(1)} \cos\theta + D_1^{(1)} \sin\theta) - K^2\tilde{r}^2 A_0^{(2)}, \tag{19d}$$

$$\begin{aligned}\phi_{10}^{(i)} = {} & A_0^{(5)} + B_0^{(5)} \log \tilde{r} + \sum_{n=1}^{\infty} \tilde{r}^{-n}(C_n^{(5)} \cos n\theta + D_n^{(5)} \sin n\theta) \\ & - 2K^2\tilde{r}\,(\log K + \tilde{\gamma} - \tfrac{1}{2} - \tfrac{1}{2}\pi i)\,(C_1^{(1)} \cos\theta + D_1^{(1)} \sin\theta) - K^2 A_0^{(1)} \tilde{r}^2 \\ & - K^2 B_0^{(1)} \tilde{r}^2 (\log \tilde{r} - 1) - 2K^2\tilde{r} \log(\tilde{r})\,(C_1^{(1)} \cos\theta + D_1^{(1)} \sin\theta) \\ & + K^2 \sum_{n=2}^{\infty} \frac{1}{(n-1)} \frac{1}{\tilde{r}^{n-2}} (C_n^{(1)} \cos n\theta + D_n^{(1)} \sin n\theta),\end{aligned} \tag{19e}$$

where $A_0^{(m)}$, $B_0^{(m)}$, $C_n^{(m)}$ and $D_n^{(m)}$ $(m = 1, 2, \ldots, 5; n = 1, 2, \ldots)$ are constants and $\tilde{\gamma} = 0{\cdot}57721$ (Euler's constant). Observe that the lowest-order solution behaves as $\log \tilde{r}$ as $\tilde{r} \to \infty$ and that both $\phi_{01}^{(i)}$ and $\phi_{12}^{(i)}$ are constants (actually the latter result comes from matching inner and outer solutions to all orders). Thus the lowest-order solution is a 'classical' solution to Laplace's equation in the sense that $\partial\phi_{00}^{(i)}/\partial\tilde{r}$ vanishes at infinity.

Similarly, the matched outer solution is given by

$$\begin{aligned}\phi^{(0)} = {} & [A_{01} + A_{02}(\tfrac{1}{2}\epsilon)^2 \log(\tfrac{1}{2}\epsilon) + A_{03}(\tfrac{1}{2}\epsilon)^2]\, H_0^{(1)}(KR) + (\tfrac{1}{2}\epsilon)\, H_1^{(1)}(KR) \\ & \times (A_{11} \cos\theta + B_{11} \sin\theta) + (\tfrac{1}{2}\epsilon)^2 H_2^{(1)}(KR)\,(A_{21} \cos 2\theta + B_{21} \sin 2\theta) + \ldots,\end{aligned} \tag{20}$$

where $A_{01}, A_{02}, A_{03}, A_{11}, B_{11}, A_{21}$ and B_{21} are constants and the $H_n^{(1)}$ $(n = 0, 1, 2, \ldots)$ are Hankel functions. The coefficients in the outer solution are given in terms of those in the inner solution by

$$A_{01} = \frac{\pi}{2i} B_0^{(1)}, \quad A_{02} = \frac{\pi}{2i} B_0^{(4)}, \quad A_{03} = \frac{\pi}{2i} B_0^{(5)} \tag{21a–c}$$

and

$$A_{11} = \pi i K C_1^{(1)}, \quad B_{11} = \pi i K D_1^{(1)}, \quad A_{21} = \pi i K^2 C_2^{(1)}, \quad B_{21} = \pi i K^2 D_2^{(1)}. \tag{22a–d}$$

Similarly, the constants in the inner solution are obtained from

$$A_0^{(1)} = A_{01}[1 + (2i/\pi)(\log K + \tilde{\gamma})], \tag{23a}$$

$$A_0^{(2)} = (2i/\pi)\, A_{01}, \quad A_0^{(3)} = (2i/\pi)\, A_{02}, \tag{23b, c}$$

$$A_0^{(4)} = A_{02}[1 + (2i/\pi)(\log K + \tilde{\gamma})] + (2i/\pi)\, A_{03}, \tag{23d}$$

$$A_0^{(5)} = A_{03}[1 + (2i/\pi)(\log K + \tilde{\gamma})]. \tag{23e}$$

Observe that we are matching the outer solution to the inner solution outside the jet (i.e. both of these solutions are for the quiescent region), so that in this section K^2 stands for $(K_1^+)^2$ and $\phi_{\nu\mu}^{(i)}$ is the inner solution outside the jet.

We wish to remark that the above results were obtained by a systematic application of the 'strict' rules of singular perturbation theory. In this particular example it is possible to short-circuit many of the algebraic steps that lead to the final results. All we need to do is to observe that the outer solution can be written down to any order of accuracy.† Rewriting this solution in terms of the inner variable and expanding in terms of ϵ, we find that

$$\phi^{(o)} = \sum_{n=0}^{\infty} H_n^{(1)}(KR)\,(\tfrac{1}{2}\epsilon)^n\,[A_{n1}\cos n\theta + B_{n1}\sin n\theta + \ldots] \tag{24a}$$

or
$$\phi^{(o)} = A_{01}[1+(2i/\pi)(\log\tilde{r}+\log K+\tilde{\gamma})]$$
$$-\frac{i}{\pi}\sum_{n=1}^{\infty}\frac{(n-1)!}{K^n\tilde{r}^n}(A_{n1}\cos n\theta + B_{n1}\sin n\theta) + \frac{2i}{\pi}A_{01}\log(\tfrac{1}{2}\epsilon) + \ldots \tag{24b}$$
$$= \phi_{00}^{(i)} + \log(\tfrac{1}{2}\epsilon)\,\phi_{01}^{(i)} + \ldots. \tag{24c}$$

From this equation we can immediately deduce the form of $\phi_{00}^{(i)}$ and the relationship between the near-field coefficients $A_0^{(1)}$, $B_0^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$ and the far-field coefficients A_{01}, A_{n1}, and B_{n1}. This relationship is, of course, that given by the asymptotic matching principle for $n = 1$ and 2. Furthermore it also appears that $\phi_{01}^{(i)} =$ constant and that A_{n1} and B_{n1} are proportional to $C_n^{(1)}$ and $D_n^{(1)}$. Thus part of the asymmetric outer solution is matched to the asymmetric part of the lowest-order inner solution.

5. Inner and jet solutions and results of matching pressure and particle displacement across jet boundary

The sequence of inner equations (15) for the velocity potential is also satisfied by the potential $\phi_{\nu\mu}^{(J)}$ inside the jet. Thus the lowest-order solution inside the jet satisfies

$$\tilde{\Delta}\phi_{00}^{(J)} = \delta(\tilde{r})/\tilde{r} \tag{25a}$$

and an application of Gauss' theorem shows that

$$\iint_{a}\tilde{\Delta}\phi_{00}^{(J)}\,\tilde{r}\,d\tilde{r}\,d\theta = 2\pi = \int_{\partial a}\frac{\partial\phi_{00}^{(J)}}{\partial n}\,ds, \tag{25b}$$

where a is the cross-sectional area of the jet, ∂a denotes the jet boundary and ds is the arc length along the boundary. Similarly, from (15a),

$$\iint_{\bar{a}}\tilde{\Delta}\phi_{00}^{(i)}\,\tilde{r}\,d\tilde{r}\,d\theta = 0 = \int_{\partial\bar{a}}\frac{\partial\phi_{00}^{(i)}}{\partial n}\,ds, \tag{25c}$$

where $\bar{a}$ is the annular area between the jet and a large circle of radius $\tilde{r} =$ constant. Clearly the boundary $\partial\bar{a}$ consists of two parts, namely the jet boundary and the circle. Combining (25b), (25c) and matching condition (10c) yields

$$2\pi\kappa = \int_{\text{circle}}\frac{\partial\phi_{00}^{(i)}}{\partial n}\,ds. \tag{26}$$

Finally, the contour integral in (26) is evaluated explicitly using (19a):

$$B_0^{(1)} = \kappa. \tag{27a}$$

† The author is grateful to one of the referees for pointing out this degeneracy of the asymptotic matching. See also Landau & Lifshitz (1959, p. 283).

By a completely analogous argument we find that

$$B_0^{(4)} = (2/\pi)\,\kappa a[(K_1^{\mp})^2 \mp \kappa^2 \rho_{12}(K_2^{\mp})^2] \tag{27b}$$

and

$$B_0^{(5)} = \frac{2}{\pi}\left[(K_1^{\mp})^2 \iint_a \phi_{00}^{(i)}\,\tilde{r}\,d\tilde{r}\,d\theta \mp \kappa(K_2^{\mp})^2 \iint_a \phi_{00}^{(J)}\,\tilde{r}\,d\tilde{r}\,d\theta\right]. \tag{27c}$$

Note that the interface matching conditions (10) were applied for each coefficient of the gauge functions. In particular, $\phi_{01}^{(i)}$ = constant outside the jet implies that $\phi_{01}^{(J)}$ = constant inside the jet.

The remarkable outcome of this asymptotic expansion is that the outer field, to the required order, depends only on the lowest-order inner and jet solutions. Both of these solutions satisfy Laplace's equation.

6. Calculation of the acoustic power of the source

The acoustic power of the source is calculated by integrating the product of the pressure and the normal component of the velocity over a large cylinder enclosing the jet. The mathematical technique is fully described by Morse & Ingard (1968, p. 728).

Let us express the velocity potential in the far field as

$$\phi^{(o)} = \sum_n H_n^{(1)}(k_0 K_1^{+} r)\,(A_n \cos n\theta + B_n \sin n\theta). \tag{28a}$$

Then comparison of (28a) and (20) shows that

$$A_0 = A_{01} + A_{02}(\tfrac{1}{2}\epsilon)^2 \log(\tfrac{1}{2}\epsilon) + A_{03}(\tfrac{1}{2}\epsilon)^2, \tag{28b}$$

$$A_1 = (\tfrac{1}{2}\epsilon)A_{11}, \quad B_1 = (\tfrac{1}{2}\epsilon)B_{11}, \quad A_2 = (\tfrac{1}{2}\epsilon)^2 A_{21}, \quad B_2 = (\tfrac{1}{2}\epsilon)^2 B_{21}. \tag{28c–f}$$

The coefficients A_n and B_n ($n = 0, 1, 2, \ldots$) are complex in general.

The expressions for the perturbation pressure and the radial component of the velocity are given by the x, t Fourier inverse of (28a),

$$p = -\rho_1 \frac{\partial \Phi}{\partial t} = \frac{i\rho_1}{4\pi^2 c_2^2 U} \int_{\omega_0/(1+M)}^{\omega_0/(1-M)} \exp\,[i(\omega-\omega_0)x/U - i\omega t]\,\omega \sum_n H_n^{(1)}(A_n \cos n\theta + B_n \sin n\theta)\,d\omega, \tag{29a}$$

$$\Phi_r = \frac{k_0}{4\pi^2 c_2^2 U} \int_{\omega_0/(1+M)}^{\omega_0/(1-M)} \exp\,[i(\omega-\omega_0)x/U - i\omega t]\,K_1^{+} \sum_n H_n^{(1)\prime}(A_n \cos n\theta + B_n \sin n\theta)\,d\omega, \tag{29b}$$

and the radiative power of the source is

$$P = \frac{\rho_1}{4\pi^4 U c_2^4} \int_0^{2\pi} P_\theta\, d\theta, \tag{30a}$$

where

$$P_\theta = \int_{\omega_0/(1+M)}^{\omega_0/(1-M)} \omega\, d\omega\, \Big| \sum_n e^{-\frac{1}{2}in\pi}(A_n \cos n\theta + B_n \sin n\theta) \Big|^2. \tag{30b}$$

P is the total power and P_θ is the 'power in a plane'† whose orientation is determined by the polar angle θ. Observe that the integral for P_θ is evaluated over a frequency range defined by the Doppler limits.

We now discuss very briefly the Kelvin–Helmholtz instability. This classic problem has been examined by a number of authors, including Miles (1957),

† P_θ is also proportional to the power per unit polar (or azimuthal) angle θ.

Batchelor & Gill (1962) and more recently, Jones & Morgan (1972). The conclusion of Jones & Morgan, although only qualitatively relevant to the present problem, is that the long-term solution for a harmonically pulsating source (switched on at $t = 0$) is a harmonically varying (in time) acoustic field which contains an additive element that grows exponentially with distance downstream but decays exponentially as away from the jet boundary. In fact, the correct solution to the present problem can be written in the form $\Psi = \phi + \phi_1$, where ϕ is obtained from our analysis and ϕ_1 is the unstable solution. Our approach at this point is to ignore the unstable contribution to the acoustic field. Of course, this approach cannot be justified mathematically. On the other hand, the physical justification is clear, since the actual jet is reasonably stable with a bounded acoustic field. It should be noted, however, that there are Mach number limitations on this physical justification (Ffowcs Williams 1973).

Additively unstable or unbounded solutions arise quite frequently in applied mathematics and their presence is usually attributed to problem idealization. Perhaps a relevant and classic example is the theory of normal hyperbolic equations, where the fundamental solution is singular on the 'Mach cone' (i.e. on the characteristic surfaces). Thus integrals of the fundamental solution over surfaces, or regions of space, are divergent in the classical sense. These difficulties were encountered by Volterra and Hadamard, who devised special (but mathematically not justified) techniques for 'evaluating' divergent integrals.

This concludes the formal analysis of the paper. We now apply the above results to two specific jet configurations. In the following examples the notation is self-contained and departs slightly from that in the main body of the paper.

7. Circular jet: radiation from off-axis sources

Mani (1972) has examined theoretically the radiation from sources convecting along the centre-line of a circular jet. We now use the present asymptotic theory to extend his results to the off-axis case in the low frequency limit.

The geometry of the jet is shown in figure 3. The jet boundary is given by $R = 1$; the source is located at $R = R_0 < 1$, $\Theta = \Theta_0$.

The lowest-order inner and jet solutions obey Laplace's equation [see (15*a*) and (25*a*)]. We represent these solutions by $\phi^{(1)} = \phi^{(i)}_{00}$ and $\phi^{(2)} = \phi^{(J)}_{00}$ respectively. Thus

$$\phi^{(1)} = A_0 + B_0 \log R + \sum_{n=1}^{\infty} R^{-n}(C_n \cos n\Theta + D_n \sin n\Theta), \tag{31a}$$

$$\phi^{(2)} = \alpha_0 + \log r + \sum_{n=1}^{\infty} R^{n}(\alpha_n \cos n\Theta + \beta_n \sin n\Theta), \tag{31b}$$

where A_0, B_0, α_0, C_n, D_n, α_n and β_n $(n = 1, 2, \ldots)$ are constants. The jet solution may be rewritten by using a well-known Fourier expansion of $\log r$ for $R > R_0$:

$$\phi^{(2)} = \alpha_0 + \log R + \sum_{n=1}^{\infty}\left[\alpha_n R^n - \frac{1}{n}\left(\frac{R_0}{R}\right)^n \cos n\Theta_0\right]\cos n\Theta + \sum_{n=1}^{\infty}\left[\beta_n R^n - \frac{1}{n}\left(\frac{R_0}{R}\right)^n \sin n\Theta_0\right]\sin n\Theta. \tag{31c}$$

The constants $A_0, B_0, \ldots$, etc., are determined by matching the pressure and particle displacement across the jet boundary $R = 1$. The results of this matching are

$$B_0 = \kappa, \tag{32a}$$

$$C_n = -\frac{2}{n}\kappa\frac{R_0^n \cos n\Theta_0}{1+\rho_{12}\kappa^2}, \quad n = 1, 2, \ldots, \tag{32b}$$

$$D_n = -\frac{2}{n}\kappa\frac{R_0^n \sin n\Theta_0}{1+\rho_{12}\kappa^2}, \quad n = 1, 2, \ldots, \tag{32c}$$

where $\kappa = \omega/\omega_0$ and $\rho_{12} = \rho_1/\rho_2$.

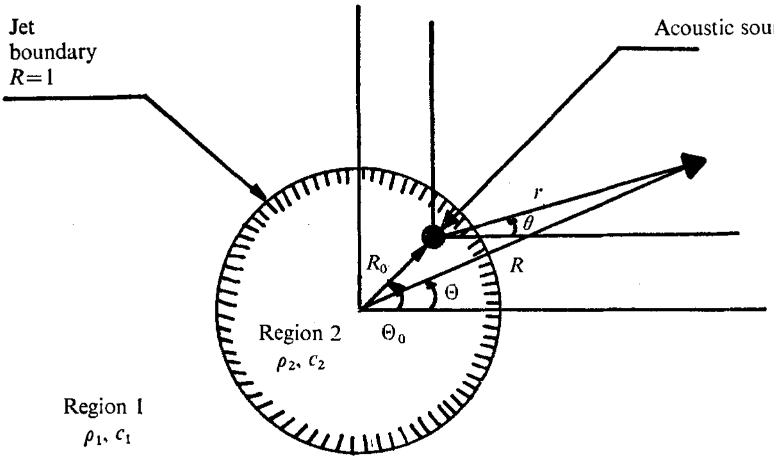

FIGURE 3. Geometry of circular jet.

Observe that, as $r \to \infty$, the co-ordinate systems (R, Θ) and (r, θ) are related by

$$r = R[(1-R_0/R)\cos(\Theta-\Theta_0)+\ldots], \tag{33a}$$

$$\theta = \Theta+(R_0/R)\sin(\Theta-\Theta_0)+\ldots \tag{33b}$$

and

$$R = r[1-(r_0/r)\cos(\theta-\theta_0)+\ldots], \tag{33c}$$

$$\Theta = \theta+(r_0/r)\sin(\theta-\theta_0)+\ldots, \tag{33d}$$

where $r_0 = R_0$ and $\theta_0 = \Theta_0+\pi$. (r_0, θ_0) denote the co-ordinates of the jet axis relative to the co-ordinate system attached to the source. After using the co-ordinate transformation (33) in (31a), we find that as $r \to \infty$

$$\begin{aligned}\phi^{(1)} = {} & A_0+B_0\log r - B_0[(r_0/r)\cos(\theta-\theta_0)+\tfrac{1}{2}(r_0/r)^2\cos 2(\theta-\theta_0)+\ldots] \\ & + C_1 r^{-1}\cos\theta + D_1 r^{-1}\sin\theta + r^{-2}\cos(2\theta)\,(C_2+C_1 r_0\cos\theta_0 - D_1 r_0 \sin\theta_0) \\ & + r^{-2}\sin(2\theta)\,(D_2+C_1 r_0\sin\theta_0+D_1 r_0\cos\theta_0)+\ldots.\end{aligned} \tag{34}$$

Thus the far-field coefficients (21) and (22) are given by (27) and (34) as

$$A_{01} = (\pi/2i)\,\kappa, \tag{35a}$$

$$A_{02} = -i\pi\kappa[(K_1^{\pm})^2 \mp \kappa^2\rho_{12}(K_2^{\pm})^2], \tag{35b}$$

$$A_{03} = \pm\tfrac{1}{2}\pi i\kappa(K_2^{\pm})^2 r_0^2 + \tfrac{1}{2}\pi i\kappa[(K_1^{+})^2 \mp (K_2^{\pm})^2] - \tfrac{1}{2}\pi^2\kappa[1+(2i/\pi)(\log K_1^{+} + \tilde{\gamma})][(K_1^{+})^2 \mp \kappa^2\rho_{12}(K_2^{\pm})^2], \tag{35c}$$

$$A_{11} = \pi i K_1^{+}\kappa r_0 \cos\theta_0(2/\Omega^{+}-1), \tag{35d}$$

$$B_{11} = \pi i K_1^{+}\kappa r_0 \sin\theta_0(2/\Omega^{+}-1), \tag{35e}$$

$$A_{21} = \tfrac{1}{2}\pi i(K_1^{+})^2\kappa r_0^2 \cos(2\theta_0)\,\Omega^{-}/\Omega^{+}, \tag{35f}$$

$$B_{21} = \tfrac{1}{2}\pi i(K_1^{+})^2\kappa r_0^2 \sin(2\theta_0)\,\Omega^{-}/\Omega^{+}, \tag{35g}$$

where $\Omega^{\pm} = 1 \pm \rho_{12}\kappa^2.$

Before we present the numerical results for the total power P, we briefly discuss some possible non-uniformities in the asymptotic expansion. It may be seen from figure 2(*a*) that K_1^{+} is of order unity; therefore the harmonic terms $A_{11}, \ldots, B_{21}$ exhibit no non-uniformities. On the other hand, it may be seen from figure 2(*b*) that terms proportional to $K_2^{\pm}$ may become much larger than unity. In this case the asymptotic expansion has a small range of validity restricted to very small values of ϵ. To extend this range, we first observe that the non-uniformity comes from the axisymmetric terms. An examination of the structure of the complete axisymmetric solution reveals that the non-uniformity arises from a binomial expansion. To eliminate this difficulty, we replace the constant A_0, given by (28*b*), by

$$A_0' = A_{01}^2/(A_{01} - A_{02}(\tfrac{1}{2}\epsilon)^2\log(\tfrac{1}{2}\epsilon) - A_{03}(\tfrac{1}{2}\epsilon)^2). \tag{36}$$

In the limit as $\epsilon \to 0$, (36) and (28*b*) are clearly equivalent. However, for finite values of ϵ, (36) has a much wider range of validity than (28*b*) in this particular example.

In figure 4 we show a comparison between the exact calculations by Mani and the results of the present asymptotic theory for the total radiative power (30*a*). The power is shown to be a function of the convective Mach number and the source Strouhal number. It may be seen that the agreement between the two results is excellent over the entire Mach number range (up to 0·9) and a wide range of Strouhal numbers.

Since the far field, to the required order of accuracy, varies as $\sin n\theta$ and $\cos n\theta$ ($n = 0, 1, 2,$) it may appear that the off-axis results are valid only in the case when the eccentricity associated with the problem is small, that is, when $R_0 \ll 1$. This point of view was expressed by one of the referees. The author thinks that there is no reason to place a bound on R_0 since the asymptotic solution does not exhibit a non-uniformity as $R_0 \to 1$. Thus the present analysis not only extends the axisymmetric results of Landau & Lifshitz (1959, p. 283) to higher order but also includes the effects of asymmetry.

By differentiating the expression (34) for the velocity potential with respect to the source co-ordinates it is a simple matter to derive the pressure and velocity field of an off-axis quadrupole shielded by a circular jet. For example, $\partial^2\phi^{(1)}/\partial r_0 \partial\theta_0$ is the velocity potential of an $r-\theta$ quadrupole, whose radiated power can be readily obtained.

A complete physical interpretation of these results is given by Mani (1972). We re-emphasize, however, that the above calculations show the frequency dependence of convective amplification. It is known from experiments (Lush 1971) that the data for the sound pressure level as a function of directivity angle (at constant source frequency) are under (over) estimated at low (high) frequencies by the Lighthill theory. We interpret this to mean that convective amplification is frequency dependent. Furthermore, the results in figure 4 show such dependence for the total power. Of course, the connexion between the results of Lush (1971) and those in figure 4 is qualitative in nature.

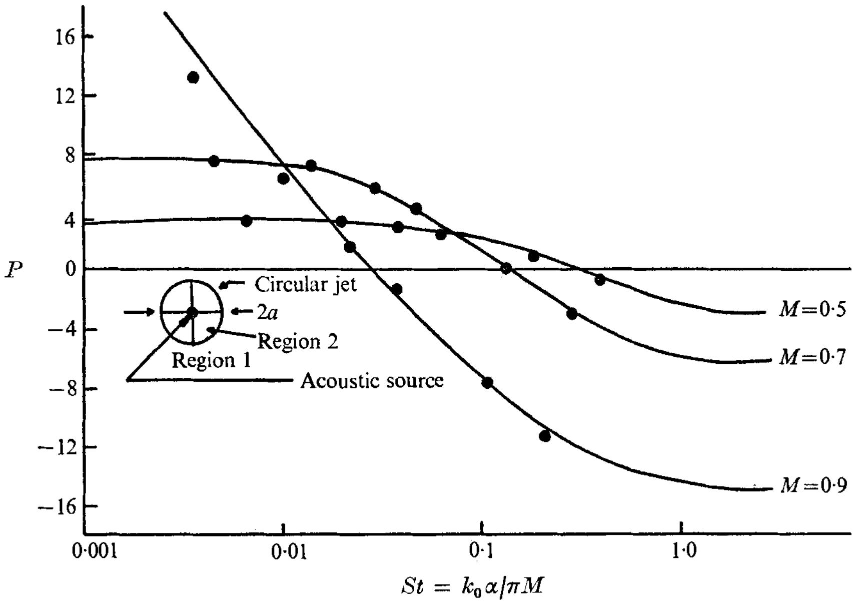

FIGURE 4. Convective amplification as function of M and St. $P = 20\log_{10}$ [(radiated power of shielded source)/(radiated power of unshielded source)]. $c_1 = c_2$, $\rho_1 = \rho_2$, M = source convective Mach number, $\lambda = 2\pi/k_0$ = wavelength of source radiation. ——, exact results of Mani (1972); ●, present asymptotic theory.

An extremely important point is that acoustic shielding *is* confined to the refractive zone of silence, but shielding is more than just refraction since the former affects the radiated power. In other words, measurements outside the zone of silence would show negligible shielding. However, such measurements are insufficient for the computation of the total power. Thus, to look at the effects of fluid shrouding or shielding, the sound pressure level must be obtained in the zone of silence.

8. Elliptic jet: radiation from axial source

Let us consider an elliptic jet whose semi-axes are given by α and β, where $\alpha \geqslant \beta \geqslant 0$. The source is located at the centre of the ellipse. We introduce elliptic co-ordinates (μ, ν) $(0 \leqslant \mu < \infty, 0 \leqslant \nu < 2\pi)$ by the transformation

$$r = \tfrac{1}{2}a(\cosh^2\mu - \sin^2\nu)^{\frac{1}{2}}, \quad 0 \leqslant r, \tag{37a}$$

$$\theta = \tan^{-1}(\tanh\mu\tan\nu), \quad 0 \leqslant \theta < 2\pi, \tag{37b}$$

where $a = 2(\alpha^2-\beta^2)^{\frac{1}{2}}$ is the focal length and (r,θ) are polar co-ordinates (figure 5). The equation of the ellipse is

$$\mu = \mu_1 = \log[(\alpha+\beta)/(\alpha^2-\beta^2)^{\frac{1}{2}}].$$

The lowest-order inner and jet solutions obey Laplace's equation [see (15*a*) and (25*a*)]. We represent these solutions by $\phi^{(1)} = \phi^{(i)}_{00}$ and $\phi^{(2)} = \phi^{(J)}_{00}$ respectively. Thus

$$\phi^{(1)} = A_0 + B_0\mu + \sum_{n=1}^{\infty} C_n e^{-n\mu}\cos n\nu, \qquad (38a)$$

$$\phi^{(2)} = \alpha_0 + \left(\mu+\log\frac{a}{4}\right) - 2\sum_{n=1}^{\infty}\frac{1}{n}e^{-n\mu}\cos n\nu\cos\frac{n\pi}{2} + \sum_{n=1}^{\infty}\alpha_n\sinh n\mu\cos n\nu, \quad (38b)$$

where A_0, B_0, α_0, C_n and α_n $(n = 1, 2, \ldots)$ are constants, again to be determined by matching the pressure and particle displacement across the jet boundary. Observe that the terms in (38*b*) that do not involve any of these unknown constants represent $\log r$ in elliptic co-ordinates.

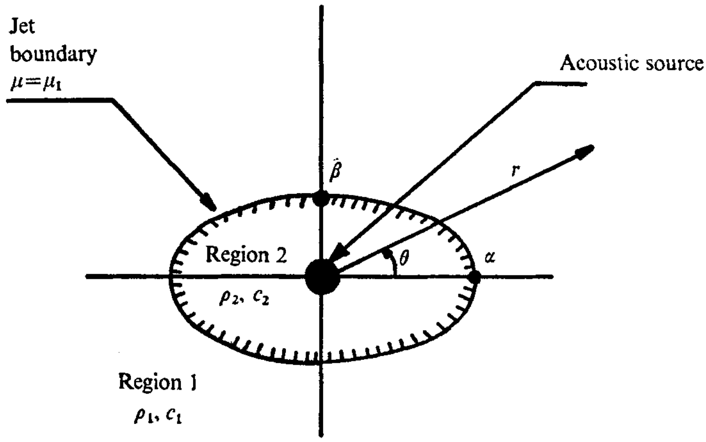

FIGURE 5. Geometry of elliptic jet.

The matching across the jet boundary yields

$$B_0 = \kappa, \quad C_n = -\frac{4}{n}\kappa\frac{\cos\frac{1}{2}n\pi}{\Omega^+ - \Omega^-\exp(-2n\mu_1)}, \quad n = 1, 2, \ldots, \qquad (39a, b)$$

and
$$\alpha_1 = 0, \quad \alpha_2 = -\frac{2}{\Omega^+/\Omega^- - \exp(-4\mu_1)}\exp(-4\mu_1), \qquad (39c, d)$$

where, as before, $\kappa = \omega/\omega_0$, $\rho_{12} = \rho_1/\rho_2$ and $\Omega^{\pm} = 1 \pm \rho_{12}\kappa^2$.

The transformation (37) to elliptic co-ordinates simplifies to

$$\mu = \log\frac{4r}{a} - \left(\frac{a}{4}\right)^2\frac{\cos 2\theta}{r^2} + \ldots, \qquad (40a)$$

$$\nu = \theta + \left(\frac{a}{4}\right)^2\frac{\sin 2\theta}{r^2} + \ldots \qquad (40b)$$

as $r \to \infty$ and, in the same limit, we find that

$$e^{-n\mu}\cos n\nu = (a/4r)^2 \cos n\theta + n(a/4r)^{n+2}\cos(n+2)\theta + \ldots, \quad n = 1, 2, \ldots. \tag{40c}$$

Using (40) in (38*a*) and the general results of the asymptotic theory (21), (22) and (27) we find that the far-field coefficients are given by

$$A_{01} = (\pi/2i)\,\kappa, \tag{41a}$$

$$A_{02} = -i\pi\alpha\beta\kappa[(K_1^+)^2 \mp \rho_{12}\kappa^2(K_2^{\pm})^2], \tag{41b}$$

$$A_{03} = -i[(K_1^+)^2 t_1 \mp \kappa(K_2^{\pm})^2 t_2], \tag{41c}$$

where

$$t_1 = \pi\alpha\beta\kappa\left\{\log\frac{\alpha+\beta}{2} - \frac{1}{2}\frac{\beta}{\alpha} - \frac{\pi i}{2}\left[1 + \frac{2i}{\pi}(\log K_1^+ + \tilde{\gamma})\right] - (1-\beta/\alpha)\right.$$
$$\left.\times\left[\Omega^+ - \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2\Omega^-\right]^{-1}\right\}, \tag{41d}$$

$$t_2 = \pi\alpha\beta\left\{\rho_{12}\kappa^2 \mathrm{los}\frac{\alpha+\beta}{2} - \frac{1}{2}\frac{\beta}{\alpha} - \frac{\pi i}{2}\kappa^2\rho_{12}\left[1 + \frac{2i}{\pi}(\log K_1^+ + \tilde{\gamma})\right]\right.$$
$$\left.-\frac{1}{2}\left(1-\frac{\beta}{\alpha}\right)\left(\frac{\Omega^+}{\Omega^-} - \frac{\alpha-\beta}{\alpha+\beta}\right)\Big/\left[\frac{\Omega^+}{\Omega^-} - \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2\right]\right\}, \tag{41e}$$

$$A_{11} = B_{11} = 0, \tag{41f}$$

$$A_{21} = \frac{\pi i}{4}(K_1^+)^2\kappa(\alpha^2-\beta^2)\left[1 + \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2\right]\Big/\left[\frac{\Omega^+}{\Omega^-} - \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2\right], \tag{41g}$$

$$B_{21} = 0. \tag{41h}$$

The asymptotic expansion appears to be uniformly valid for all values of $\beta \leqslant \alpha$. Thus it seems again that this low frequency theory is not restricted to configurations with 'low departure from axisymmetry'.

The power P_θ [see (30*b*)] in a plane inclined at an angle θ to the major axis is calculated as a function of θ for various values of $\epsilon = k_0\alpha(\alpha = 1)$ and β. The results are shown in figure 6(*a*) and (*b*) for two different source Mach numbers. For given values of θ, β and M, say $\theta = 0$, $\beta = 0{\cdot}5$ and $M = 0{\cdot}7$, the radiative power of the source decreases with increasing source frequency. The power for a given geometry at a fixed value of θ and for moderate values of ϵ decreases with increasing Mach number. These observations are consistent with the results in figure 4 for a circular jet.

The difference in acoustic power between the quiet plane ($\theta = 0$) and the noisy plane ($\theta = 90°$) increases with the source Strouhal number and Mach number. According to our calculations the difference in power between the two planes is completely negligible at low frequencies and is of the order of a few db at higher frequencies. These conclusions agree qualitatively with the low velocity experimental findings of Olsen *et al.* (1973).

The total radiative power (30*a*) of the source varies inversely with the jet cross-sectional area at given Strouhal and Mach numbers. This observation is easily deducible from the results of figure 6 and agrees with the fluid shielding hypothesis of Mani. In simple terms, the radiative efficiency of the source varies inversely with the amount of moving fluid surrounding it.

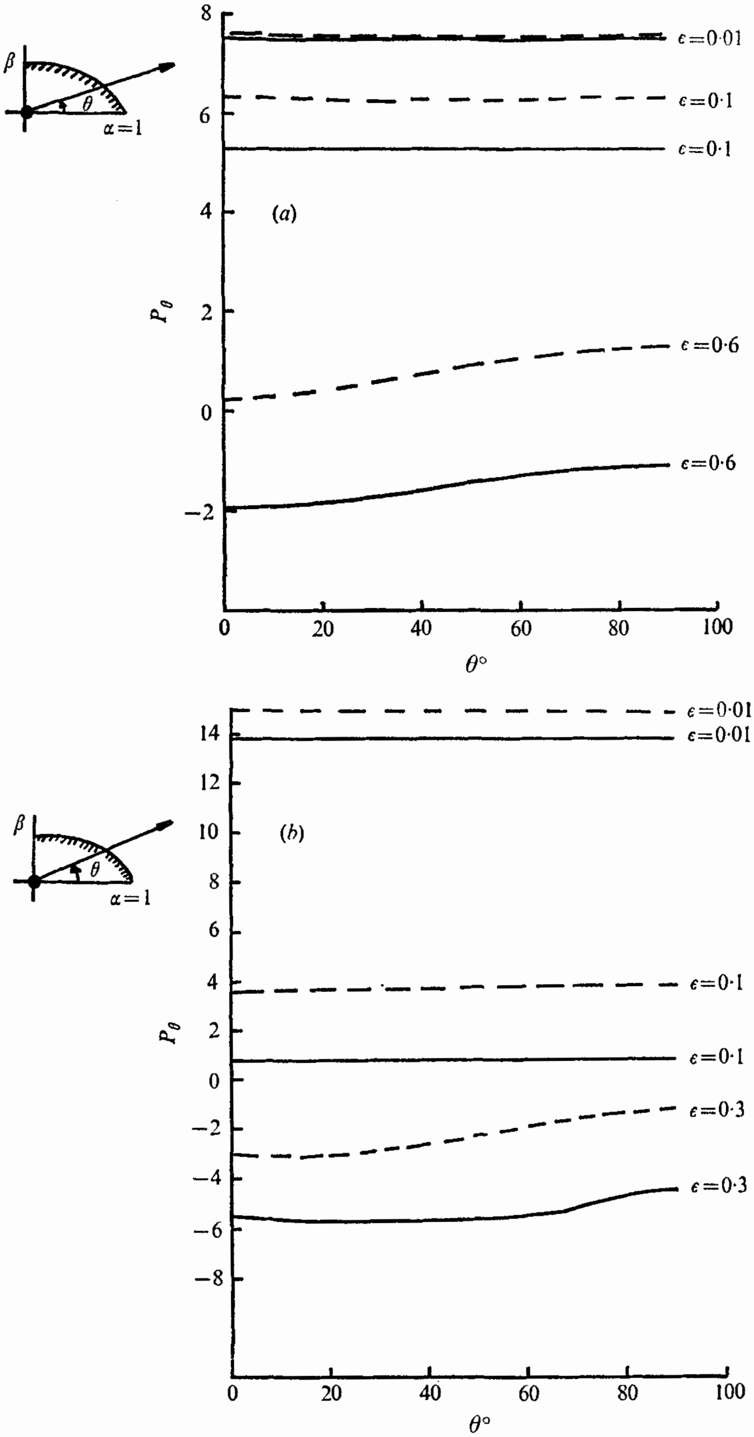

FIGURE 6. Theoretical power calculations for elliptic jet. $P_\theta = 20 \log_{10}$ [(power per unit θ of shielded source)/(power per unit θ of unshielded source)]. $c_1/c_2 = 1{\cdot}0$, $\rho_1/\rho_2 = 1{\cdot}0$, $\epsilon = k_0\alpha$. ——, $\beta = 0{\cdot}5$; – – –, $\beta = 0{\cdot}2$. (*a*) $M = 0{\cdot}7$. (*b*) $M = 0{\cdot}9$.

9. Conclusions

It has been shown that in the low frequency limit the inner and outer solutions obey the Poisson and Helmholtz equations respectively. The appropriate inner length scale is the jet diameter and the outer length scale is the wavelength. The outer solution, to order $(\frac{1}{2}\epsilon)^2$, depends only on the lowest-order inner and jet solutions. The error in the outer field is $O(\epsilon^3 \log \epsilon)$.

The asymptotic results for the circular jet indicate that the present theory is accurate for values of ϵ up to 0·7 or 0·8. The non-uniformity of the expansion associated with the limit $M \to 1$ (ϵ fixed) can be eliminated, for the most part, by re-expressing the coefficient of the axisymmetric part of the far field as a fraction.

As $\epsilon \to 0$ the power of the jet becomes independent of jet shape. This conclusion appears plausible since in this limit all jets appear from the far field as 'thin lines'.

The results for the elliptic jet offer one qualitative explanation for the presence and location of certain experimentally observed quiet planes. This explanation is purely acoustic and centres around the shrouding effect of the mean flow.

The financial support for this study was provided by the U.S. Department of Transportation. I should like to thank Dr R. Mani for suggesting the problem and for many valuable discussions in the area of jet noise.

REFERENCES

BATCHELOR, G. K. & GILL, A. E. 1962 *J. Fluid Mech.* **14**, 529.
CRIGHTON, D. G. 1973 *J. Fluid Mech.* **59**, 665.
CSANADY, G. T. 1966 *J. Fluid Mech.* **26**, 183.
FRAENKEL, L. E. 1969 *Proc. Camb. Phil. Soc.* **65**, 209.
FFOWCS WILLIAMS, J. E. 1973 AGARD-CPP-131 (Noise Mechanisms), 1–1.
GERMAIN, P. 1967 *J. Roy. Aero. Soc.* **71**, 677.
HOCH, R. & HAWKINS, R. 1973 AGARD-CPP-131 (Noise Mechanisms), 19–1.
JONES, D. S. & MORGAN, J. D. 1972 *Proc. Camb. Phil. Soc.* **72**, 465.
LANDAU, L. D. & LIFSHITZ, E. M. 1959 *Fluid Mechanics*. Addison-Wesley.
LIGHTHILL, M. J. 1952 *Proc. Roy. Soc.* A **211**, 564.
LILLEY, G. M. 1972 AFAPL-TR-72-53, vol. 4.
LUSH, P. A. 1971 *J. Fluid Mech.* **46**, 477.
MANI, R. 1972 *J. Sound Vib.* **25**, 337.
MANI, R. 1974 *J. Fluid Mech.* **64**, 611.
MILES, J. W. 1953 *J. Acoust. Soc. Am.* **25**, 1087.
MILES, J. W. 1957 *J. Acoust. Soc. Am.* **29**, 226.
MORSE, P. M. & INGARD, K. U. 1968 *Theoretical Acoustics*. McGraw-Hill.
OLSEN, W. A., GUTIERREZ, O. A. & DORSCH, R. G. 1973 *N.A.S.A. Tech. Memo.* X-68-182.
PHILLIPS, O. M. 1960 *J. Fluid Mech.* **9**, 1.
POWELL, A. 1960 *J. Acoust. Soc. Am.* **32**, 1609.
RIBNER, H. S. 1957 *J. Acoust. Soc. Am.* **29**, 435.
RIBNER, H. S. 1960 *J. Acoust. Soc. Am.* **32**, 1159.
RIBNER, H. S. 1962 *U.T.I.A.S. Rep.* no. 86.
VAN DYKE, M. 1964 *Perturbation Methods in Fluid Mechanics*. Academic.